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Inverting the Shapovalov Form

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We provide an explicit formula for the inversion of the Shapovalov form associated to irreducible Verma modules. This formula implies that the singularities of the inverse are all simple poles. © 1992 Academic Press, Inc.

A basic tool in understanding Verma modules is the associated $U(\mathbf{g})$-invariant bilinear pairing, the Shapovalov form. For example, Shapovalov's determinant formula provides a criterion for the irreducibility of Verma modules [1]. In the work of Goodman and Wallach [2], the Shapovalov form was explicitly inverted for special functionals. The convergence properties of this inverse were then used to deduce dual analyticity results.

Here we explicitly describe the entire inverse of the Shapovalov form. As a surprising consequence, we show that the singularities are all simple, rather than the order of Kostant's partition function, the naive estimate arising from Shapovalov's determinant formula. We shall apply this explicit inverse in a future work to construct dual analytic functionals analogous to Whittaker vectors.

To begin we establish some notation. Let $\mathbf{g}$ be a semisimple Lie algebra over $\mathbf{C}$, with a nondegenerate symmetric invariant form $\langle \cdot, \cdot \rangle$, and let $\mathbf{h}$ be a Cartan subalgebra of $\mathbf{g}$. Then $\langle \cdot, \cdot \rangle$ restricts to a nondegenerate form on $\mathbf{h} \times \mathbf{h}$. For $\alpha \in \mathbf{h}^*$, let $H_\alpha \in \mathbf{h}$ be defined by $\langle H_\alpha, H \rangle = \alpha(H)$ for $H \in \mathbf{h}$. Let $\langle \cdot, \cdot \rangle$ also denote the $\mathbf{C}$-linear form on $\mathbf{h}^* \times \mathbf{h}^*$ induced by $\langle \cdot, \cdot \rangle$. In particular, $\langle \alpha, \beta \rangle := \langle H_\alpha, H_\beta \rangle$. Let Φ be the root system of $(\mathbf{g}, \mathbf{h})$. Adopt the convention that $0 \in \Phi$. Fix Φ^+ $(0 \notin \Phi^+)$, a system of positive roots, and let Δ be the simple roots for Φ^+. Let

$$N\Delta := \left\{ \sum n_\alpha \alpha : n_\alpha \in \mathbf{N}, \alpha \in \Delta \right\},$$

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where $\mathbf{N} := \{0, 1, 2, ...\}$. Also use the corresponding notation for $\mathbf{Z}$. Define a partial ordering on $\mathbf{Z}\Delta$:

$$\mu \preccurlyeq \nu \Leftrightarrow \nu - \mu \in \mathbf{N}\Delta. \tag{1}$$

Let

$$\mathbf{u} := \sum_{\alpha \in \Phi^+} \mathbf{g}^\alpha, \qquad \bar{\mathbf{u}} := \sum_{\alpha \in \Phi^+} \mathbf{g}^{-\alpha}.$$

Then

$$\mathbf{g} = \bar{\mathbf{u}} \oplus \mathbf{h} \oplus \mathbf{u}.$$

From the nondegeneracy of $\langle \cdot, \cdot \rangle$, it is possible to select $X_\alpha \in \mathbf{g}^\alpha$, $X_{-\alpha} \in \mathbf{g}^{-\alpha}$ for $\alpha \in \Phi^+$ so that $\langle X_\alpha, X_{-\beta} \rangle = \delta_{\alpha\beta}$.

To simplify notation, fix an enumeration $\{\alpha_1, ..., \alpha_l\}$ of Φ^+. For a multi-index $I = (i_1, ..., i_m)$ with $1 \leqslant i_j \leqslant l$, let $\mu_I := \sum_{j=1}^m \alpha_{i_j}$ and $-I := (-i_m, ..., -i_1)$ (note the inversion of the ordering). Also write X_i for X_{α_i}. Here $\alpha_{-j} := -\alpha_j$. Then

$$X_I := X_{i_1} \cdots X_{i_m} \qquad \text{and} \qquad X_{-I} := X_{-i_m} \cdots X_{-i_1}.$$

Allow $I = \varnothing$ with the conventions that $0^0 := 1$, $\mu_\varnothing := 0$, and $X_\varnothing = X_{-\varnothing} := 1$. The corresponding truncated multi-indices are

$$I(k) := (i_k, ..., i_m).$$

Similarly, for a multi-exponent $K = (k_1, ..., k_l) \in \mathbf{N}^l$, let $\mu^K := \sum_{i=1}^l k_i \alpha_i$ and

$$X^K := X_1^{k_1} X_2^{k_2} \cdots X_l^{k_l} \qquad \text{and} \qquad X^{-K} := X_{-l}^{k_l} \cdots X_{-2}^{k_2} X_{-1}^{k_1}.$$

Also let $I(\mu)$ be the collection of all multi-indices I such that $\mu = \mu_I$ and $K(\mu)$ be the collection of all multi-exponents K such that $\mu = \mu^K$.

Next we recall the definition of a Verma module. For $\Lambda \in \mathbf{h}^*$, let $\mathbf{C}_\Lambda = \mathbf{C} \cdot 1_\Lambda$ be the one dimensional $\mathbf{h}$-module such that

$$H \cdot 1_\Lambda = \Lambda(H) 1_\Lambda$$

for $H \in \mathbf{h}$. Give $\mathbf{C}_\Lambda^*$ the canonical dual right $\mathbf{h}$-module structure. Provide $\mathbf{C}_\Lambda$ with a left $(\mathbf{h} + \mathbf{u})$-module structure via $(H + X) \cdot 1_\Lambda := H \cdot 1_\Lambda$ for $H \in \mathbf{h}$, $X \in \mathbf{u}$. Select $1_\Lambda^* \in \mathbf{C}_\Lambda^*$ so that $1_\Lambda^*(1_\Lambda) = 1$ and provide $\mathbf{C}_\Lambda^*$ with a right $(\mathbf{h} + \bar{\mathbf{u}})$-module structure via $1_\Lambda^* \cdot (H + Y) := 1_\Lambda^* \cdot H := 1_\Lambda^* \Lambda(H)$ for $H \in \mathbf{h}$, $Y \in \bar{\mathbf{u}}$. Define the *Verma modules*

$$\begin{aligned} M_\Lambda &:= U(\mathbf{g}) \otimes_{U(\mathbf{h}+\mathbf{u})} \mathbf{C}_\Lambda \cong_{U(\bar{\mathbf{u}})} U(\bar{\mathbf{u}}) \otimes_{\mathbf{C}} \mathbf{C}_\Lambda, \\ \bar{M}_\Lambda &:= \mathbf{C}_\Lambda^* \otimes_{U(\mathbf{h}+\bar{\mathbf{u}})} U(\mathbf{g}) \cong_{U(\mathbf{u})} \mathbf{C}_\Lambda^* \otimes_{\mathbf{C}} U(\mathbf{u}). \end{aligned} \tag{2}$$

The *Shapovalov form* is the canonical $U(\mathbf{g})$-invariant pairing $\langle\cdot,\cdot\rangle_\Lambda : \bar{M}_\Lambda \times M_\Lambda \to \mathbf{C}$ defined via the maps

$$\bar{M}_\Lambda \times M_\Lambda \to \bar{M}_\Lambda \otimes_{U(\mathbf{g})} M_\Lambda \cong \mathbf{C}_\Lambda^* \otimes_{U(\mathbf{h})} \mathbf{C}_\Lambda \to \mathbf{C},$$

where $\mathbf{C}_\Lambda^* \otimes_{U(\mathbf{h})} \mathbf{C}_\Lambda \to \mathbf{C}$ is the evaluation $e^* \otimes e \mapsto e^*(e)$.

Let K_Λ ($\bar{K}_\Lambda$) be the right (left) radical of $\langle\cdot,\cdot\rangle_\Lambda$. Then the irreducible quotients of the Verma modules are

$$L_\Lambda := M_\Lambda / K_\Lambda, \qquad \bar{L}_\Lambda := \bar{K}_\Lambda / \bar{M}_\Lambda.$$

Let $\langle\cdot,\cdot\rangle_\Lambda$ also denote the nondegenerate $U(\mathbf{g})$-invariant form induced on $\bar{L}_\Lambda \times L_\Lambda$. For $f \in (\bar{L}_\Lambda^{\Lambda-\mu})^*$, choose $v^f \in L_\Lambda^{\Lambda-\mu}$ so that

$$f(u) = \langle u, v^f \rangle_\Lambda \qquad \forall u \in \bar{L}_\Lambda^{\Lambda-\mu}. \tag{3}$$

The key observation for inverting the Shapovalov form is the following simple lemma.

LEMMA 1. *If $f \in (\bar{L}_\Lambda^{\Lambda-\mu})^*$ and $p \in U(\mathbf{u})^\mu$, then $p \cdot v^f = f(1_\Lambda^* \cdot p) 1_\Lambda$.*

Proof. Since $p \cdot v^f \in L_\Lambda^\Lambda$ and $L_\Lambda^\Lambda = \mathbf{C}1_\Lambda$ is one dimensional with $\langle 1_\Lambda^*, 1_\Lambda \rangle_\Lambda = 1$, it is only necessary to consider $\langle 1_\Lambda^*, p \cdot v^f \rangle_\Lambda = \langle 1_\Lambda^* \cdot p, v^f \rangle_\Lambda = f(1_\Lambda^* \cdot p)$. ∎

Let $\{H_1, ..., H_d\}$ be a basis for $\mathbf{h}$ so that $\langle H_i, H_j \rangle = \delta_{ij}$. Let $C \in U(\mathbf{g})$ be the Casimir element of $U(\mathbf{g})$ corresponding to $\langle\cdot,\cdot\rangle$. For the chosen basis of $\mathbf{g}$, $\{X_\alpha : \alpha \in \Phi^+\} \cup \{X_{-\alpha} : \alpha \in \Phi^+\} \cup \{H_1, ..., H_d\}$, the corresponding dual basis (relative to $\langle\cdot,\cdot\rangle$) is simply $\{X_{-\alpha} : \alpha \in \Phi^+\} \cup \{X_\alpha : \alpha \in \Phi^+\} \cup \{H_1, ..., H_d\}$. Hence

$$C = \sum_{i=1}^{d} H_i^2 + \sum_{\alpha \in \Phi^+} X_\alpha X_{-\alpha} + \sum_{\alpha \in \Phi^+} X_{-\alpha} X_\alpha.$$

Let

$$\rho := \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

Then

$$C = \sum_{i=1}^{d} H_i^2 + 2H_\rho + 2 \sum_{\alpha \in \Phi^+} X_{-\alpha} X_\alpha. \tag{4}$$

Let V be a quotient of M_Λ. We easily see that C acts on V by $\langle \Lambda, \Lambda + 2\rho \rangle$. As an immediate consequence, we have

LEMMA 2 [2, p. 228, (4.3)]. *Let* $v \in V^{\Lambda-\mu}$. *Then*

$$p(\Lambda, \mu)v = \sum_{\alpha \in \Phi^+} X_{-\alpha} X_\alpha \cdot v,$$

where

$$p(\Lambda, \mu) := \tfrac{1}{2}\langle 2(\Lambda+\rho)-\mu, \mu\rangle.$$

In order to apply Lemma 2 inductively, we need notation to describe certain families of multi-indices. First, we provide a partial ordering for multi-indices:

$$J \preccurlyeq I \Leftrightarrow \exists k \ni J = I(k). \tag{5}$$

Let $\mathbf{I}$ be a finite set of multi-indices. Then $\mathbf{I}$ is said to be a *tree* of multi-indices if $\mathbf{I}$ has the following properties:

1. $I \in \mathbf{I}$, $J \preccurlyeq I \Rightarrow J \in \mathbf{I}$. (In particular, $\varnothing \in \mathbf{I}$.)
2. If $I \in \mathbf{I}$ and if J is a multi-index so that $J(2) = I(2)$, then $J \in \mathbf{I}$.

Let the maximal multi-indices in $\mathbf{I}$, $\max \mathbf{I}$, be the set of $I \in \mathbf{I}$ so that there does not exist $J \in \mathbf{I}$ with $J \succ I$. We say that $\mathbf{I}$ is (Λ, μ)-*permissible* if $\mathbf{I}$ is a tree such that:

- If $I \in \mathbf{I} - \max \mathbf{I}$, then $p(\Lambda, \mu - \mu_I) \neq 0$. (In particular, $\mu \neq \mu_I$ for any $I \in \mathbf{I} - \max \mathbf{I}$.)

For any multi-index, define

$$p_I(\Lambda, \mu) := p(\Lambda, \mu)\, p(\Lambda, \mu - \mu_{I(|I|)}) \cdots p(\Lambda, \mu - \mu_{I(2)}), \tag{6}$$

with $p_\varnothing(\Lambda, \mu) := 1$. We now generalize a result of Goodman and Wallach [2, p. 229, Corollary 4.3].

THEOREM 3 (Recursion Formula). *Let* $v \in V^{\Lambda-\mu}$. *If* $\mathbf{I}$ *is a* (Λ, μ)-*permissible tree, then*

$$v = \sum_{I \in \max \mathbf{I}} p_I(\Lambda, \mu)^{-1} X_{-I} X_I \cdot v.$$

Proof. The result easily follows by induction from Lemma 2. ∎

Now let

$$\Omega' := \{\Lambda \in \mathbf{h}^* : p(\Lambda, \mu) \neq 0\ \forall \mu \in \mathbf{N}\Delta - \{0\}\}.$$

If $\Lambda \in \Omega'$ and $\mu \in \mathbf{N}\Delta - \{0\}$, then $p_I(\Lambda, \mu) \neq 0$ for any $I \in I(\mu)$. In particular, the nondegeneracy of the Shapovalov form for the irreducible quotients of Verma modules implies that M_Λ is irreducible for $\Lambda \in \Omega'$. Consider the (Λ, μ)-permissible tree $\mathbf{I}$ defined as follows. Let

$$\mathbf{J} := \{J : \mu_J \preccurlyeq \mu\} \supset I(\mu).$$

Though not a tree, $\mathbf{J}$ is easily expanded into one by letting $\mathbf{I}$ be the smallest tree containing $\mathbf{J}$:

$$\mathbf{I} := \{I : \exists J \in \mathbf{J} \text{ with } I(2) = J(2)\} \cup \{\emptyset\}.$$

Clearly $I \in \max \mathbf{I}$ implies that either $I \in I(\mu)$ or $\mu_I \not\preccurlyeq \mu$ (and hence $X_I \cdot v = 0$). By Theorem 3, this proves:

Corollary 4. *Let $\Lambda \in \Omega'$ and $v \in V^{\Lambda - \mu}$. Then*

$$v = \sum_{I \in I(\mu)} p_I(\Lambda, \mu)^{-1} X_{-I} X_I \cdot v.$$

For $v \in M_\Lambda^{\Lambda - \mu}$, let $S_{-\mu}(\Lambda)(v)(\cdot) := \langle \cdot, v \rangle_\Lambda$. For M_Λ irreducible and $f \in (\bar{M}_\Lambda^{\Lambda - \mu})^*$, we then have $v^f = S_{-\mu}(\Lambda)^{-1}(f)$ and from Lemma 1,

Corollary 5. *Let $\Lambda \in \Omega'$. For $f \in (\bar{M}_\Lambda^{\Lambda - \mu})^*$,*

$$S_{-\mu}(\Lambda)^{-1}(f) = \sum_{I \in I(\mu)} p_I(\Lambda, \mu)^{-1} f(1_\Lambda^* \cdot X_I) X_{-I} \cdot 1_\Lambda.$$

Since $\{X^K : \mu = \mu^K\}$ and $\{X^{-K} : \mu = \mu^K\}$ are bases for $U(\mathbf{u})^\mu$ and $U(\bar{\mathbf{u}})^{-\mu}$, respectively, the Shapovalov form $S_{-\mu}(\Lambda)$ is defined by the matrix entries

$$(\langle 1_\Lambda^* \cdot X^K, X^{-L} \cdot 1_\Lambda \rangle_\Lambda)_{K, L},$$

where K, L range over the multi-exponents in $K(\mu)$.

Theorem 6 (Shapovalov [1]; cf. [3, p. 103, Theorem 1]).

$$\det S_{-\mu}(\Lambda) = a_{-\mu} \prod \{p(\Lambda, j\alpha)^{P(\mu - j\alpha)} : \alpha \in \Phi^+, j > 0 \text{ and } j\alpha \preccurlyeq \mu\},$$

where $P(\cdot)$ is the Kostant partition function, and $a_{-\mu}$ is a constant independent of Λ.

In order to discuss the order of singularities, we first need to show that $p(\cdot, \mu)$ defines different primes as μ varies over non-zero elements of $N\Delta$.

Lemma 7. *Let $\mu, \nu \in N\Delta - \{0\}$. If there exists $c \in \mathbf{C}$ such that $p(\cdot, \mu) = cp(\cdot, \nu)$, then $\mu = \nu$.*

Proof. The main issues are the nondegeneracy of $\langle \cdot, \cdot \rangle$ and its positive definiteness on $\mathrm{N}\Delta$ (cf. [4, p. 440]). ∎

Shapovalov's formula together with Lemma 7 implies that the matrix entries of $S_{-\mu}(\Lambda)^{-1}$ (relative to the bases $\{X^{-L} \cdot 1_\Lambda\}$ and $\{1_\Lambda^* \cdot X^K\}$) can have poles only "at" multiples of the positive roots. But Corollary 5 implies that the poles of these matrix entries are all simple. Thus Lemma 1 permits us to extend a result of Goodman and Wallach from Whittaker vectors [2, p. 231, Lemma 4.4] to the entire Shapovalov form.

THEOREM 8. *The linear map*

$$S_{-\mu}(\Lambda)^{-1} \cdot \prod_{\substack{\alpha \in \Phi^+, j > 0 \\ j\alpha \preccurlyeq \mu}} p(\Lambda, j\alpha)$$

is holomorphic on $\mathbf{h}^*$.

In closing, we remark that all of these results can easily be extended to Kac–Moody Lie algebras associated with symmetrizable Cartan matrices. The reader is referred to the discussion of Kac and Kazhdan [3] for details concerning the Shapovalov form for Kac–Moody Lie algebras. The key issue is the existence of a nondegenerate symmetric invariant form and of a Casimir operator on Verma modules (cf. [3, p. 99, Lemma 1.1 and p. 102, Lemma 2.1]).

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